Supplementary Material for 'Space-time modelling of extreme events'

Raphaël Huser* and A. C. Davison*

March 18, 2013

1 Theoretical asymptotic relative efficiencies

In §3 of the paper, we introduce our maximum pairwise likelihood estimator $\widehat{\psi}_{p,\mathcal{K}}$ for spatio-temporal extremes, and in §4, we conduct a simulation study in one dimension to assess its statistical efficiency properties with respect to different schemes of time lag inclusion. Here, we add some analytical results about the asymptotic relative efficiency of similar estimators for AR(1) and MA(1) time series models. Our results confirm those obtained by Davis and Yau (2011). However, unlike Davis and Yau (2011), our objective was to understand how the asymptotic relative efficiency depends on the set of time lags $\mathcal{K}$ used to select pairs in the likelihood. In particular, our results shed some light on the question of the choice of lags when a fixed number K is predetermined. Complementary results on the efficiency of pairwise likelihood may be found in Cox and Reid (2004), Varin and Vidoni (2009), Hjort and Varin (2008) or Joe and Lee (2009).

Figure 1 displays the asymptotic relative efficiency (ARE) of the pairwise likelihood estimator with respect to the maximum likelihood estimator, that is $\mathrm{avar}(\widehat{\psi}_{\mathrm{MLE}})/\mathrm{var}(\widehat{\psi}_{p,\mathcal{K}})$, for different sets $\mathcal{K}$ of time lags. As in §4, we consider (a) $\mathcal{K}_a^K=\{1,\ldots,K\}$, for which all time lags are used up to some maximum time lag K; (b) $\mathcal{K}_b^K=\{b_k:k=1,\ldots,K\}$, where b_k is based on the Fibonacci sequence; and (c) $\mathcal{K}_c^K=\{2^{k-1}:k=1,\ldots,K\}$, where the lags increase geometrically. Since the efficiency curves were found to be qualitatively similar for (b) and (c), the figure shows only the results for $\mathcal{K}_a^K$ and $\mathcal{K}_c^K$. The left-hand column of Figure 1 displays the ARE for the AR(1) process, and the results for the MA(1) process are shown in the right-hand column. The top row shows the efficiency curves for $\mathcal{K}_a^K$ and the bottom row considers the set $\mathcal{K}_c^K$. As mentioned by Davis and Yau (2011), the efficiency is maximized when pairs at lag 1 only are included.

*Ecole Polytechnique Fédérale de Lausanne, EPFL-FSB-MATHAA-STAT, Station 8, 1015 Lausanne, Switzerland. Raphael.Huser@epfl.ch, Anthony.Davison@epfl.ch

In the top left panel, for AR(1) and $\mathcal{K}_a^K$, the ARE for the dependence parameter λ is 100% when $\mathcal{K} = \{1\}$ and then decreases sharply before stabilizing at about lag 9. This shape is reproduced qualitatively in the bottom left panel, when only the pairs at lags 2^k are taken into account, but the efficiency stabilizes at a higher level. However, in practice, one might need to include more distant pairs to ensure parameter identifiability. When the pairs at lags $1, 2, 3, 4, 5, 6$ are included, the efficiency of the estimator, around 70%, is significantly lower than when the pairs at lags $1, 2, 4, 8, 16, 32$ are included. Therefore, for a fixed number of pairs, here 6, it is best to include some distant pairs as well. This corroborates the simulation-based findings of §4 for the max-stable Schlather model with random sets.

The efficiencies for the MA(1) process are little affected either by the selection of pairs or by the number of time lags considered, but the ARE is extremely low for the dependence parameter λ. Other results (not shown) reveal that the efficiency for λ drops dramatically as λ approaches ± 1, so the loss in ARE is substantial even for moderately correlated MA(1) processes.

When max-stable processes are considered, these results can only be treated as analogies. However, it seems that two main conclusions can be drawn: including many pairs in the pairwise likelihood can spoil the estimator, suggesting that we should retain as few pairs as possible, provided the parameters remain identifiable; and if further pairs are to be used in addition to adjacent ones, estimation for autoregressive-type processes is least damaged by including temporally distant (or weakly correlated) pairs.

2 Additional plots for the simulation study

In §4 of the paper, we conduct a simulation study in dimension one to assess the efficiency properties of our pairwise likelihood estimator defined in §3.1. The simulation study is based on 1000 replications of the Schlather model with a random set, transformed to the Student t_5 scale, so that exceedances over the 95th empirical percentile are approximately GPD(σ, ξ) with shape parameter $\xi = 0.2$. The transformed data are then multiplied by the factor 2.1065 so that $\sigma \approx 1.6$ at the 95%-quantile. These parameters were chosen to mimic the rainfall data analysed in our application in §5. The top panel of Figure 2 displays a realization from this model, and the bottom panel shows how the bias and variance of the estimator depend on sample size.

3 Sliced pairwise likelihoods

In §5 of the paper, we fit the spatio-temporal Schlather model with random set to a precipitation dataset recorded in Switzerland. The random set plays the role of storms in the space-time domain, and we model it as a tilted cylinder with mean radius m_R, mean duration m_D and mean velocity

$\{\|V\| \cos(\nu), \|V\| \sin(\nu)\}^T$; see §5.2. In Figure 3, the scaled pairwise likelihood is plotted against $\|V\|$, ν, m_D and m_R. We can see that the pairwise likelihood is highly right-skewed with respect to these parameters, except for the angle ν. Moreover, for m_D it is almost flat over the region $m_D > 40$ hr. This may explain why it is difficult to fit the model to the data and to estimate the random set parameters.

References

Cox, D. R. and Reid, N. (2004) A note on pseudolikelihood constructed from marginal densities. *Biometrika* **91**(3), 729–737.

Davis, R. A. and Yau, C. Y. (2011) Comments on pairwise likelihood in time series models. *Statistica Sinica* **21**, 255–277.

Hjort, N. L. and Varin, C. (2008) ML, PL, QL in Markov chain models. *Scandinavian Journal of Statistics* **35**(1), 64–82.

Joe, H. and Lee, Y. (2009) On weighting of bivariate margins in pairwise likelihood. *J. Multivar. Anal.* **100**(4), 670–685.

Varin, C. and Vidoni, P. (2009) Pairwise likelihood inference for general state space models. *Econometric Reviews* **28**(1-3), 170–185.

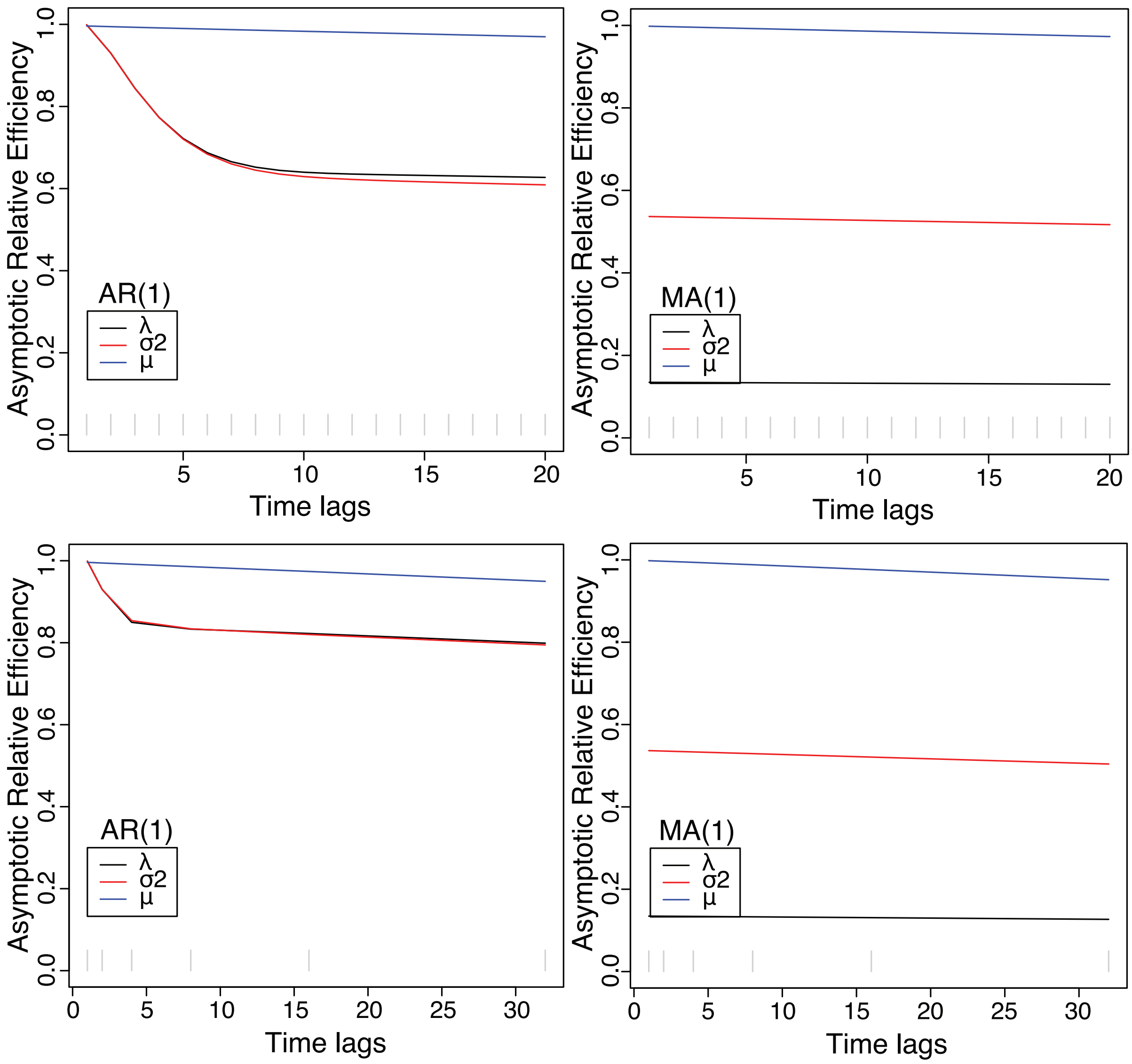

Figure 1: Asymptotic efficiency of maximum pairwise likelihood estimators relative to the maximum likelihood estimator, as a function of the maximum time lag included in the pairwise likelihood. The pairwise likelihood of equation (8) is modified by setting $S = 1$ and replacing p_u by the corresponding pairwise density. *Top row*: $\mathcal{K}_a^K = \{1, \dots, K\}$. *Bottom row*: $\mathcal{K}_c^K = \{2^{k-1}; k = 1, \dots, K\}$. *Left column*: AR(1) process $(Z_t - \mu) = \lambda(Z_{t-1} - \mu) + \varepsilon_t$, with $\varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$, $\sigma > 0, \mu \in \mathbb{R}, |\lambda| < 1$. *Right column*: MA(1) process $Z_t = \mu + \varepsilon_t + \lambda\varepsilon_{t-1}$, with $\varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma^2)$, $\sigma > 0, \mu \in \mathbb{R}, |\lambda| < 1$. The parameters are $\lambda = 0.6$, $\sigma^2 = 1$, and $T = 500$.

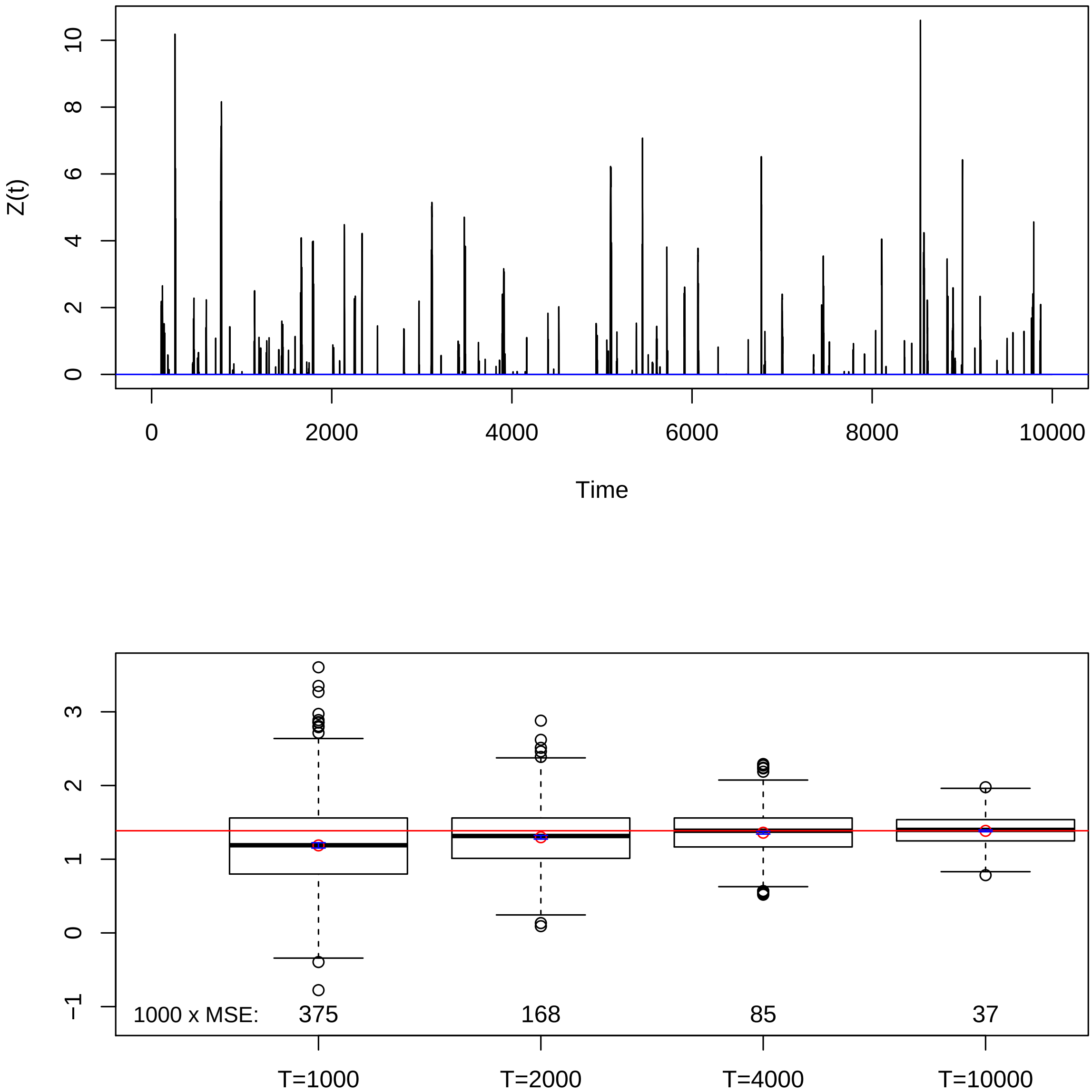

Figure 2: Simulated extreme rainfall process. *Top*: Exceedances over the 95th percentile from a simulation of the Schlather model at a particular location with beta distributed random sets. The marginal distribution of the exceedances is approximately $\mathrm{GPD}(1.6, 0.2)$ and the correlation of the underlying Gaussian random process is exponential with range parameter $\lambda = 4$, giving an effective range of 12 within random sets. *Bottom*: Boxplots (and mean squared errors) of the estimates of $\log \lambda$ (based on 1000 replications) using pairs at lag 1 only, for an increasing number of observations T. The one-step estimator was used, fixing the random set parameters to their true values. The true value is shown by the horizontal line, and the average estimate is shown by the circle close to it.

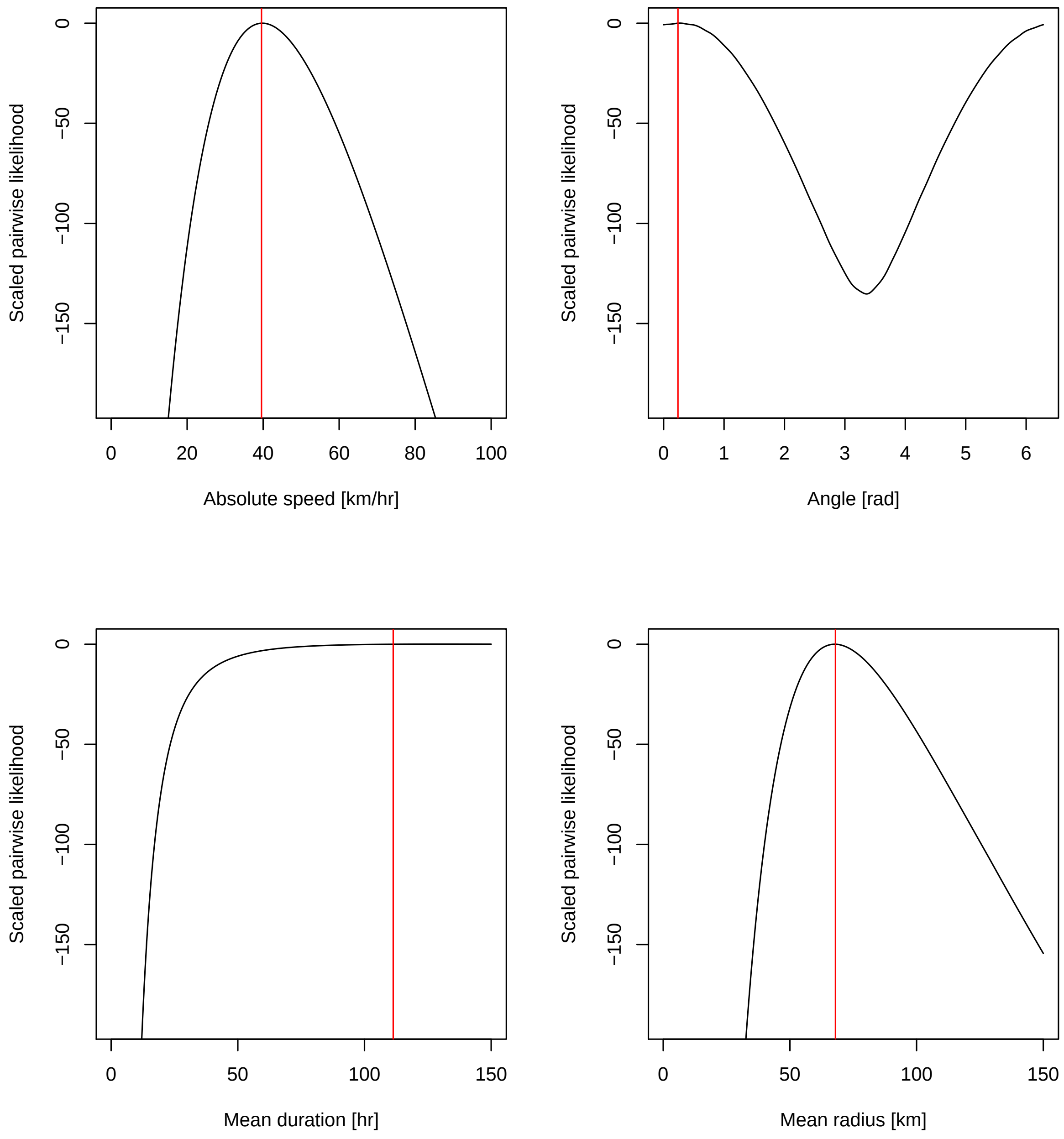

Figure 3: Sliced pairwise likelihood around the maximum pairwise likelihood estimator, with respect to the mean absolute speed $\|V\|$ (top left) and mean angle ν (top right) of the dominant winds, mean duration m_D (bottom left) and mean radius m_R (bottom right) of a storm, shifted and scaled in such a way that its value is comparable to the likelihood under independence. The vertical red lines show the maximum pairwise likelihood estimates.